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The Basis Number of the n-Cube

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The *basis number* of a graph G is defined as the least integer k such that G has a k-fold basis for its cycle space. MacLane (*Fund. Math.* **28** (1937), 22–32) has shown that a graph is planar if and only if its basis number is $\leqslant 2$. The basis numbers of certain classes of nonplanar graphs have been previously investigated. The basis number of the n-cube for every n is determined in the paper.

1. INTRODUCTION

Throughout this paper we consider only finite, undirected graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated. For undefined terms see [1].

Let $e_1, e_2,..., e_q$ be an ordering of the edges in a connected graph G. Then any subset S of edges corresponds to a (0, 1)-vector $(a_1, a_2,..., a_q)$ in the usual way, with $a_i = 1$ if $e_i \in S$ and $a_i = 0$, otherwise. These vectors form a q-dimensional vector space over the field Z_2.

Those vectors corresponding to the cycles in G generate a subspace of $(Z_2)^q$ called the *cycle space* of G, and denoted by $\mathscr{C}(G)$. (We shall say, however, that the cycles themselves, rather than the vectors corresponding to the cycles, generate $\mathscr{C}(G)$.) It is well known that the dimension of $\mathscr{C}(G)$ is $q - p + 1$, where p and q denote, respectively, the number of vertices and edges in G. In fact, given any spanning tree T in G, every graph $T + e$, $e \notin T$, contains exactly one cycle C_e, and this collection of cycles $\{C_e | e \notin T\}$ forms a basis of $\mathscr{C}(G)$, termed the *fundamental basis corresponding to T.* While each edge outside of T occurs in exactly one cycle of this basis, an edge of T

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itself may occur in many cycles of the basis. This observation suggests the following:

DEFINITION. A basis of $\mathscr{C}(G)$ is called *k-fold* if each edge of G occurs in at most k of the cycles in the basis. The *basis number* of G (denoted by $b(G)$) is the smallest integer k such that $\mathscr{C}(G)$ has a k-fold basis.

MacLane [2] proved that a graph G is planar if and only if $b(G) \leqslant 2$. In [3], the basis numbers of certain important classes of nonplanar graphs (i.e., complete graphs, complete bigraphs, and n-cubes) were investigated. (The n-cube, of course, is the graph Q_n whose vertices are the ordered n-tuples of 0's and 1's, two vertices being adjacent if and only if they differ in exactly one coordinate. Note that Q_n has 2^n vertices, $n2^{n-1}$ edges, girth 4, and is n-regular.) It was shown in [3] that $b(Q_n) \leqslant n-1$ for all n, and that $b(Q_n) \geqslant 4$ for $n \geqslant 8$. It was conjectured that $b(Q_n) = 4$ for all $n \geqslant 8$.

The purpose of this paper is to present a proof of this conjecture. Indeed, we shall establish the basis number of Q_n for every n.

2. MAIN RESULTS

We begin with a result which provides a lower bound for the basis number of a graph.

THEOREM 1. *For any connected graph G,*

$$\sum_{v \in G} \left\lfloor \frac{b(G)\, d(v)}{2} \right\rfloor \geqslant \text{girth}\, G \dim \mathscr{C}(G),$$

where $d(v)$ denotes the degree of vertex v.

Proof. Let B be a $b(G)$-fold basis of $\mathscr{C}(G)$. Now, each vertex v of G is a vertex in at most $\lfloor b(G)\, d(v)/2 \rfloor$ cycles contained in B (since each edge incident to v occurs in at most $b(G)$ cycles of B and each cycle containing v uses exactly two edges incident to v). Thus the sum of the lengths of the cycles of B is at most $\sum_{v \in G} \lfloor b(G)\, d(v)/2 \rfloor$.

The sum of the lengths of the cycles of B, however, is at least girth G $\dim \mathscr{C}(G)$. Thus we obtain

$$\sum_{v \in G} \lfloor b(G)\, d(v)/2 \rfloor \geqslant \text{girth}\, G \dim \mathscr{C}(G). \quad ∎$$

COROLLARY. *For $n \geqslant 7$, $b(Q_n) \geqslant 4$.*

Proof. Suppose that $b(Q_n) \leqslant 3$. Applying Theorem 1, we must have

$$\sum_v \left\lfloor \frac{3n}{2} \right\rfloor \geqslant 4(n2^{n-1} - 2^n + 1), \text{ or } 2^n \left\lfloor \frac{3n}{2} \right\rfloor \geqslant 2^{n+1}(n-2) + 4,$$

or

$$\left\lfloor \frac{3n}{2} \right\rfloor \geqslant 2n - 4 + 1/2^{n-2}.$$

But this is true if and only if $n \leqslant 6$. Hence if $n \geqslant 7$, we have $b(Q_n) \geqslant 4$. ∎

We now show that $b(Q_n) \leqslant 4$ for all n by constructing a 4-fold basis for any Q_n.

For each $n \geqslant 1$, let us define a sequence $\tau_n = \langle t(n, 1), t(n, 2), ..., t(n, 2^n - 1)\rangle$ recursively as follows: Take $\tau_1 = \langle 1 \rangle$, and for $n > 1$, define τ_n by

$$\begin{aligned} t(n, i) &= t(n-1, k), && \text{if } \ i = 2k, \\ &= n, && \text{otherwise.} \end{aligned}$$

(For example, $\tau_2 = \langle 2, 1, 2 \rangle$ and $\tau_3 = \langle 3, 2, 3, 1, 3, 2, 3 \rangle$.) It is easily seen that the sequences τ_n have the following properties:

(i) If $n \geqslant 3$ and $t(n, i) = t(n, j) \leqslant n - 1$, there exist an *even* number of terms with value n between $t(n, i)$ and $t(n, j)$;

(ii) If $n \geqslant 3$ and $t(n, i) \leqslant n - 2$, there exist an *even* number of terms with value n preceeding $t(n, i)$.

We now use τ_n to define a Hamiltonian path in Q_n. Let $W_n = w_1 w_2 \cdots w_{2^n}$ be the path with $w_1 = (1, 1, ..., 1, 0)$ and where w_i and w_{i+1} differ in exactly the $t(n, i)$ coordinate. It is readily verified by induction that W_n is indeed a Hamiltonian path.

Just before giving the construction of our 4-fold basis for $\mathscr{C}(Q_n)$, we need a bit of notation. We shall refer to the subgraph of Q_n in which the vertices correspond to the ordered n-tuples whose nth coordinate is 0 (resp., 1) as Q'_{n-1} (resp., Q''_{n-1}). It is easily seen that Q'_{n-1} and Q''_{n-1} are isomorphic to Q_{n-1}.

Let B_n be a collection of 4-cycles in Q_n defined recursively as follows: Take $B_2 = \{Q_2\}$, and for $n > 2$ let $B_n = B'_{n-1} \cup B''_{n-1} \cup S_{n-1}$, where B'_{n-1} (B''_{n-1}) is the subset of 4-cycles in Q'_{n-1} (Q''_{n-1}) corresponding to the set B_{n-1} in Q_{n-1}, and S_{n-1} is the set of 4-cycles $C_i = w'_i w'_{i+1} w''_{i+1} w''_i w'_i$, for $i = 1, 2, ..., 2^{n-1} - 1$, where w'_i, w'_{i+1} (w''_i, w''_{i+1}) are consecutive vertices of W'_{n-1} (W''_{n-1}).

LEMMA 1. *B_n is a basis of $\mathscr{C}(Q_n)$.*

Proof. The lemma is true when $n = 2$, and we proceed by induction on n. Since B'_{n-1} and B''_{n-1} are bases for $\mathscr{C}(Q'_{n-1})$ and $\mathscr{C}(Q''_{n-1})$ respectively, each of them contains $\dim \mathscr{C}(Q_{n-1}) = (n-3)\,2^{n-2} + 1$ distinct cycles. Also, since W'_{n-1} and W''_{n-1} both contain $2^{n-1} - 1$ distinct edges, S_{n-1} contains $2^{n-1} - 1$ distinct 4-cycles. Hence we obtain

$$|B_n| = |B'_{n-1}| + |B''_{n-1}| + |S_{n-1}| = n2^{n-1} - 2^n + 1 = \dim \mathscr{C}(Q_n).$$

Thus to show that B_n is a basis of $\mathscr{C}(Q_n)$, it suffices to verify that the 4-cycles of B_n are independent.

Suppose that there exists a subset of cycles in B_n, say $B \subseteq B_n$, which satisfies a nontrivial relation modulo 2 (that is, $\sum_{C \in B} C = 0 \pmod 2$). Since B'_{n-1} and B''_{n-1} are both bases, and no cycle in B'_{n-1} has an edge in common with any cycle in B''_{n-1}, it follows that B must include at least one cycle C_i in S_{n-1}. But then it follows easily that $C_1 = w'_1 w''_1 w''_2 w'_2 w'_1$ is an element of B. (For C_i contains the edge $w'_i w''_i$. The only other cycle in B_n containing this edge is C_{i-1}. Hence, C_{i-1} must belong to B to cancel $w'_i w''_i$ modulo 2. Continuing this argument, one concludes that B must contain C_1.) But the 4-cycle C_1 contains the edge $w'_1 w''_1$ which occurs in no other 4-cycle of B_n, and in particular in no other cycle of B. This means that $\sum_{C \in B} C$ could not be 0 modulo 2, a contradiction. Therefore, B_n is a basis of $\mathscr{C}(Q_n)$. ∎

An edge e in Q_n is said to be of *type* k if and only if the vertices of e differ in exactly the kth coordinate. Two edges e_1 and e_2 of type $k \neq l$ are said to be *l-correspondent* if and only if the two vertices of e_2 can be obtained by changing the lth coordinate of the vertices of e_1.

LEMMA 2. *B_n is at most 4-fold.*

Proof. Lemma 2 is easily verified for $n = 2$ and 3. Hence, we proceed by induction on $n \geqslant 4$.

By hypothesis, B'_{n-1} and B''_{n-1} are at most 4-fold, and the edges joining Q'_{n-1} to Q''_{n-1} appear in at most two 4-cycles of S_{n-1}. It suffices to show that the edges of W'_{n-1} and W''_{n-1} are at most 3-fold in B'_{n-1} and B''_{n-1}, respectively. By symmetry, we need show this only for W'_{n-1}.

Let $e'_{j,k,n-1}$ denote the jth edge of type k in the Hamiltonian path W'_{n-1}. If $k \geqslant n-2$, clearly, $e'_{j,k,n-1}$ could not have been used in four previous cycles. Suppose therefore that $1 \leqslant k \leqslant n-3$. Consider $e'_{j,k,n-2}$ in W'_{n-2}. By the inductive hypothesis, $e'_{j,k,n-2}$ occurs in at most four 4-cycles of B_{n-1} and hence of B'_{n-1}. Since it appears once in W'_{n-2}, it follows that $e'_{j,k,n-2}$ appears in at most three 4-cycles of B'_{n-2}.

Let e''_k denote the $(n-2)$-correspondent of $e'_{j,k,n-2}$. Since any two $(n-2)$-correspondent edges appear in the same number of basis cycles of B'_{n-2}, e''_k

also appears in at most three 4-cycles of B'_{n-2}. Moreover, it is easily seen (by property i) that at most one of a pair of $(n-2)$-correspondent edges can occur in W'_{n-2}. Since $e'_{j,k,n-2}$ belongs to W'_{n-2}, it follows that e''_k does not belong to W'_{n-2}. Hence, e''_k occurs in at most three 4-cycles of B'_{n-1}.

To complete the proof, we shall show that e''_k is precisely $e'_{j,k,n-1}$. (It will then follow that $e'_{j,k,n-1}$ occurs in at most three 4-cycles of B'_{n-1} as desired.) To show this, note that the coordinates of the initial vertices of W'_{n-1} and W'_{n-2} (the latter considered as a path in Q'_{n-1}) differ only in the $(n-2)$th coordinate. Now, τ_{n-1} is formed by placing a term with value $(n-1)$ in front of each term of τ_{n-2}. Since $k \leqslant n-3$, an *even* number of $(n-1)$s have been placed before any k valued term in τ_{n-1} (by property (ii)) and, in particular, before the jth occurrence of a k in τ_{n-1}. It follows easily therefore that $e'_{j,k,n-1}$ is precisely the $(n-2)$-correspondent of $e'_{j,k,n-2}$.

This completes the proof. ∎

By Lemmas 1 and 2 we have

THEOREM 2. *For all* n, $b(Q_n) \leqslant 4$.

Theorem 2 and the Corollary to Theorem 1 immediately imply

THEOREM 3. *For* $n \geqslant 7$, $b(Q_n) = 4$.

We observe that $b(Q_n) = n-1$, for $n = 2$, 3, and 4. (For $n = 2$ this is trivial. For $n = 3$ and 4, it follows by Theorem 1 and the upper bound $b(Q_n) \leqslant n-1$ given in [3].) We show finally that $b(Q_5) = b(Q_6) = 3$.

The construction of a basis of $\mathscr{C}(Q_n)$ given in Lemma 1 can be generalized as follows:

(i) the bases B'_{n-1} and B''_{n-1} need not correspond—any two bases $B_{n-1,1}$ and $B_{n-1,2}$ of $\mathscr{C}(Q_{n-1})$ can be used,

(ii) the Hamiltonian path W_{n-1} of Q_{n-1} can be replaced by any spanning tree T_{n-1}.

The basis for $\mathscr{C}(Q_n)$ which results from this generalized construction will be designated as $B_n = (B_{n-1,1}, B_{n-1,2}, T_{n-1})$.

Denote the vertices of Q_n by $v_0, v_1, \dots, v_{2^n-1}$, where $v_k = (a_1, a_2, \dots, a_n)$ if $k = \sum_{i=0}^{n-1} a_{n-i} 2^i$. In Q_3 consider the cycles $C_1 = v_0 v_1 v_3 v_2 v_0$, $C_2 = v_0 v_1 v_5 v_4 v_0$, $C_3 = v_0 v_2 v_6 v_4 v_0$, $C_4 = v_1 v_3 v_7 v_5 v_1$, $C_5 = v_4 v_5 v_7 v_6 v_4$, and $C_6 = v_2 v_3 v_7 v_6 v_2$. Then $B_{3,1} = \{C_1, C_2, C_3, C_4, C_5\}$ and $B_{3,2} = \{C_2, C_3, C_4, C_5, C_6\}$ are bases of $\mathscr{C}(Q_3)$. Next let $B_{4,1} = (B_{3,1}, B_{3,2}, T_{3,1})$ and $B_{4,2} = (B_{3,1}, B_{3,2}, T_{3,2})$, where $T_{3,1}$ (resp., $T_{3,2}$) is the Hamiltonian path $v_4 v_5 v_1 v_0 v_2 v_3 v_7 v_6$ (resp., $v_1 v_0 v_2 v_3 v_7 v_6 v_4 v_5$) in Q_3. Finally, we let $B_5 = (B_{4,1}, B_{4,2}, T_4)$ and $B_6 = (B_5, B_5, T_5)$, where T_4 and T_5 are the spanning

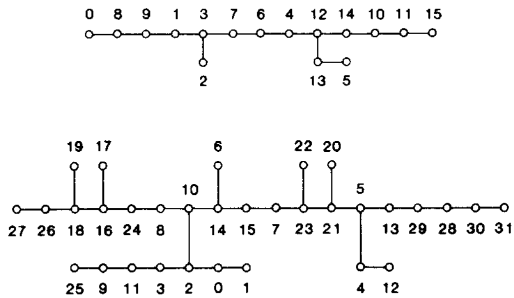

FIG. 1. T_4 and T_5.

trees in Q_4 and Q_5 shown in Fig. 1 (in which only the indices of the vertices are displayed). It can be readily verified that the bases B_5 and B_6 are indeed 3-fold.

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